

Proper identification of the gluon spin

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Abstract

Properties of the recently proposed gauge-invariant gluon spin S_g are studied and compared to the usually defined “gluon polarization” Δg . By explicit 1-loop calculations in a quark state, it is found that $S_g = \frac{5}{9}\Delta g$. Furthermore, $\frac{4}{5}$ of S_g can actually be identified as a “static-field” contribution and shown to cancel exactly an analogous static term in the gluon orbital angular momentum L_g , leaving $S_g + L_g$ unaltered. These observations suggest that if properly identified, the gluon contribution to the nucleon spin may be drastically smaller than in the conventional wisdom.

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In gauge theories, the basic physical notions such as gluon or photon spin and orbital angular momentum suffer from a severe problem with gauge invariance, because construction of these quantities necessarily involves the canonical variable of the gauge field A_μ , which includes a gauge freedom. This problem has long interfered with a coherent investigation of the nucleon spin structure, and only recently, was a solution proposed in Refs.[1, 2], which aroused considerable interest in the community [3–7]. The key to the solution is the extraction of a physical (gauge-invariant) part \hat{A}_μ out of the gauge field A_μ . (An analog of this idea actually has a long history in gravity, in the attempt to extract the true gravitational degrees of freedom out of the metric which also describes the inertial effect [8]). With this gauge-invariant field variable \hat{A}_μ , the gluon or photon spin *density* can be easily defined gauge-invariantly as $\vec{E} \times \vec{\hat{A}}$, which is the subject of close examination in this paper.

To start, we review and re-formulate the gauge-invariant construction in Refs.[1, 2] for convenient use in the present calculation. For an Abelian gauge field, the explicit expression of \hat{A}_μ is [9]:

$$\hat{A}_\mu(\vec{x}, t) = \frac{1}{\partial^2} \partial_i F_{i\mu}(\vec{x}, t) = \int d^3x' \frac{\partial'_i F_{i\mu}(\vec{x}', t)}{-4\pi|\vec{x} - \vec{x}'|} \quad (1)$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. Greek indices run from 0 to 3, Latin indices run from 1 to 3, and repeated indices are summed over (even when they both appear raised or lowered). Gauge invariance of \hat{A}_μ is evident from its expression in terms of $F_{\mu\nu}$, and it can be easily verified that

$$\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu = \frac{1}{\partial^2} (\partial_\mu \partial_i F_{i\nu} - \partial_\nu \partial_i F_{i\mu}) = F_{\mu\nu} \quad (2a)$$

$$\partial_i \hat{A}_i = \frac{1}{\partial^2} \partial_i \partial_k F_{ki} = 0, \quad (2b)$$

namely, that \hat{A}_μ has null spatial divergence, and produces the full field strength $F^{\mu\nu}$. Thus,

$$\bar{A}_\mu \equiv A_\mu - \hat{A}_\mu = A_\mu - \frac{1}{\partial^2} \partial_i F_{i\mu} \quad (3)$$

is the pure-gauge part of A_μ , and is fully responsible for the spatial divergence of A_μ :

$$\partial_\mu \bar{A}_\mu - \partial_\nu \bar{A}_\nu = 0 \quad (4a)$$

$$\partial_i \bar{A}_i = \partial_i A_i \quad (4b)$$

Equivalently, Eqs.(2) and (4) can be taken as the defining equations for \hat{A}_μ and \bar{A}_μ , with Eqs.(1) and (3) as their solutions, respectively. However, one must be careful about

boundary conditions. For the integration in Eq.(1) to be meaningful, the field strength $F_{\mu\nu}$ must vanish fast enough at infinity. It is natural to assign the same boundary condition for the physical field \hat{A}_μ ; then Eq.(1) is the unique solution to Eqs.(2), and Eq. (3) is the unique solution to Eqs.(4) under the boundary condition that \bar{A}_μ approaches A_μ at infinity. Note that A_μ or more precisely its pure-gauge part \bar{A}_μ may not vanish at infinity even if $F_{\mu\nu}$ does. However, if they do (as typically obtains in perturbative calculations), the expressions for \hat{A}_μ and \bar{A}_μ can be greatly simplified:

$$\hat{A}_\mu = \frac{1}{\bar{\partial}^2} \partial_i (\partial_i A_\mu - \partial_\mu A_i) = A_\mu - \partial_\mu \frac{1}{\bar{\partial}^2} \partial_i A_i \quad (5a)$$

$$\bar{A}_\mu = \partial_\mu \frac{1}{\bar{\partial}^2} \partial_i A_i \quad (5b)$$

For the non-Abelian gluon field, $A_\mu \equiv A_\mu^a T^a$, with T^a the color matrix, the defining equations for \hat{A}_μ and \bar{A}_μ are most elegantly arranged as [2, 9]

$$\bar{F}_{\mu\nu} \equiv \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu + ig[\bar{A}_\mu, \bar{A}_\nu] = 0, \quad (6a)$$

$$\bar{\mathcal{D}}_i \hat{A}_i \equiv \partial_i \hat{A}_i + ig[\bar{A}_i, \hat{A}_i] = 0. \quad (6b)$$

That is, \bar{A}_μ is still a pure gauge, and the *gauge covariant* spatial divergence of \hat{A}_μ vanishes. The physical field, \hat{A}_μ , (like the non-Abelian field strength $F_{\mu\nu}$) is now gauge-covariant instead of gauge-invariant, and has to be solved for perturbatively. The leading term is the same as in Eq. (1). At next-to-leading order, the expression is [9]:

$$\hat{A}_\mu = \frac{1}{\bar{\partial}^2} \partial_i F_{i\mu} + ig \frac{1}{\bar{\partial}^2} \left\{ \left[\frac{1}{\bar{\partial}^2} \partial_k F_{ki}, \partial_i \frac{1}{\bar{\partial}^2} \partial_k F_{k\mu} - \partial_i A_\mu \right] - \partial_i [A_i, \frac{1}{\bar{\partial}^2} \partial_k F_{k\mu}] + \partial_\mu [\frac{1}{\bar{\partial}^2} \partial_k F_{ki}, A_i] \right\} + \mathcal{O}(g^2) \quad (7)$$

This expression only requires that $F_{\mu\nu}$ vanish fast enough at infinity. If A_μ does also, then the expressions for \hat{A}_μ and \bar{A}_μ simplify to

$$\hat{A}_\mu = A_\mu - \bar{A}_\mu \quad (8a)$$

$$\bar{A}_\mu = \partial_\mu \frac{1}{\bar{\partial}^2} \partial_i A_i + ig \left\{ \partial_\mu \frac{1}{\bar{\partial}^2} [\partial_i \frac{1}{\bar{\partial}^2} \partial_k A_k, A_i] - \partial_i \frac{1}{\bar{\partial}^2} [\partial_i \frac{1}{\bar{\partial}^2} \partial_k A_k, \partial_\mu \frac{1}{\bar{\partial}^2} \partial_k A_k] \right\} + \mathcal{O}(g^2) \quad (8b)$$

Separation of the physical and gauge degrees of freedom in A_μ can be of great use and convenience. A significant example is the gauge-invariant, complete decomposition of the total QCD angular momentum operator into four terms [1, 2]:

$$\begin{aligned} \vec{J}_{\text{QCD}} &= \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \vec{x} \times \psi^\dagger \frac{1}{i} \vec{D} \psi + \int d^3x \vec{E} \times \vec{A} + \int d^3x \vec{x} \times E_i \vec{\mathcal{D}} \hat{A}_i \\ &\equiv \vec{S}_q + \vec{L}_q + \vec{S}_g + \vec{L}_g. \end{aligned} \quad (9)$$

Here $\bar{D}_\mu \equiv \partial_\mu + ig\bar{A}_\mu$ is the pure-gauge covariant derivative for the quark field, and $\bar{\mathcal{D}}_\mu \equiv \partial_\mu + ig[\bar{A}_\mu, \]$ is the pure-gauge covariant derivative for a field in the adjoint representation. For an Abelian theory, \hat{A}_μ is gauge invariant and does not need a covariant derivative, so the decomposition of the total QED angular momentum operator is a little simpler (in the sector of photon orbital angular momentum):

$$\begin{aligned}\vec{J}_{\text{QED}} &= \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \vec{x} \times \psi^\dagger \frac{1}{i} \vec{D} \psi + \int d^3x \vec{E} \times \vec{\hat{A}} + \int d^3x \vec{x} \times E_i \vec{\partial} \hat{A}_i \\ &\equiv \vec{S}_e + \vec{L}_e + \vec{S}_\gamma + \vec{L}_\gamma.\end{aligned}\tag{10}$$

[A remarkable (and somewhat mysterious) fact is that using the ordinary derivative can also lead to a gauge-invariant gluon orbital angular momentum, which however dictates a very specific definition of \hat{A}_μ . See Ref. [1] for a detailed discussion.]

Note that Eqs.(9) and (10) provide *operational* decompositions: Given the explicit expressions of \hat{A}_μ and \bar{A}_μ , we can straightforwardly make calculations with the operators in Eqs.(9) and (10). We will concentrate here on the gauge-invariant “gluon spin”, \vec{S}_g . In comparison, the widely discussed “gluon polarization” Δg is related to the gauge-dependent operator $\int d^3x \vec{E} \times \vec{A}$ in the light-cone gauge [10]. The aim of this paper is to investigate the properties of \vec{S}_g , and to make a quantitative comparison with Δg . To make the conclusion as concrete as possible, we take the simplest non-trivial example of a 1-loop calculation in an on-shell quark state. As we will show, S_g is typically much smaller than Δg . Moreover, a remarkable and physically appealing feature of our gauge-invariant definitions in Eq.(9) is that the “static field” does not contribute to the total gluon angular momentum, $S_g + L_g$. [The same feature holds for the total photon angular momentum, $S_\gamma + L_\gamma$]. If the static pieces are subtracted altogether from S_g and L_g (which leaves the sum, $S_g + L_g$, unaltered), the remaining “dynamic” gluon spin is only $\frac{1}{9}$ of Δg .

Since \vec{S}_g and \vec{L}_g are explicitly gauge-invariant operators, the calculation can be performed in any gauge for convenience. The covariant gauge has the simplest Feynman rules. But for our purpose the Coulomb gauge $\vec{\partial} \cdot \vec{A} = 0$ is the most convenient: Eqs.(8) indicate that as $\vec{\partial} \cdot \vec{A}$, we have $\bar{A}_\mu = 0$ and $\hat{A}_\mu = A_\mu$ in this gauge.

For the 1-loop calculation of the gluon matrix element in a quark state, the gluon field behaves like eight independent Abelian fields. Consider a quark state, $|p\sigma\rangle$, with momentum

p and polarization σ along the third axis. At 1-loop order one finds [11]

$$\Delta g \equiv \langle p\sigma | \int d^3x (\vec{E} \times \vec{A})_3 | p\sigma \rangle_{A^+=0} = \sigma \cdot 2 \frac{\alpha_s}{\pi} \ln \frac{Q^2}{m^2} \quad (11)$$

where Q^2 and m^2 are the ultraviolet and infrared cutoffs, respectively. For comparison, our gauge-invariant “gluon spin” leads to

$$\begin{aligned} S_g &\equiv \langle p\sigma | \int d^3x (\vec{E} \times \vec{\tilde{A}})_3 | p\sigma \rangle \\ &= \langle p\sigma | \int d^3x (\vec{E} \times \vec{A})_3 | p\sigma \rangle_{\vec{\partial} \cdot \vec{A}=0} = \frac{5}{9} \Delta g. \end{aligned} \quad (12)$$

Here we have used the important and convenient relation that the matrix element of the *gauge-invariant* operator $\vec{E} \times \vec{\tilde{A}}$ (in any gauge) is the same as that of the *gauge-dependent* operator $\vec{E} \times \vec{A}$ in the Coulomb gauge. (The reader should keep in mind that Δg has been defined only in light-cone gauge, whereas we have only *chosen* to evaluate our *gauge-invariant* operator in Coulomb gauge.)

We see that S_g is much smaller than Δg . The reason can be traced to the fact that S_g is constructed solely with the “physical” gluon field, while Δg is calculated in the light-cone gauge in which A_μ contains both physical and pure-gauge components; thus Δg includes a non-physical pure-gauge contribution. This suggests that S_g is a more physical and reasonable definition of the gluon spin than Δg . And indeed, Δg leads to a spuriously large gluon content in a parent quark state. A rather heuristic way to see this is by renormalizing the divergent S_g and Δg in a very specific way, namely, by choosing the ultraviolet cutoff Q^2 to be the same as the scale for the running coupling constant:

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{12\pi}{(33 - 2n_f) \ln(Q^2/\Lambda^2)}. \quad (13)$$

If the quark flavor n_f is set to 3, then at large Q^2 we find $\Delta g = \frac{8}{9}\sigma$, while $S_g \simeq 0.5\sigma$.

We now derive the major conclusion of this paper, namely that the gauge-invariant gluon or photon spin $\vec{E} \times \vec{\tilde{A}}$ can be further separated into two gauge-invariant terms, one of which, a “static-field” term, exactly cancels an analogous term in the gluon or photon orbital angular momentum. The point is that the separation $A_\mu = \hat{A}_\mu + \bar{A}_\mu$ allows one to split the electric field \vec{E} into *gauge-invariant* (in Abelian case) or *gauge-covariant* (in non-Abelian case) pieces. (Such splitting is not possible with the full A_μ .)

For the simpler Abelian case, we have:

$$\vec{E} = -\partial_t \vec{A} - \vec{\partial} A^0 = -\partial_t \vec{\tilde{A}} - \vec{\partial} \hat{A}^0 \equiv \vec{E}^{\text{dy}} + \vec{E}^{\text{st}}. \quad (14)$$

We call $\vec{E}^{\text{dy}} \equiv -\partial_t \vec{\hat{A}}$ the “dynamic” field and $\vec{E}^{\text{st}} \equiv -\vec{\partial} \hat{A}^0$ the “static” field. [We apologize that the word “static” is not very pertinent, because $-\vec{\partial} \hat{A}^0$ can as well be time-dependent. By “static” we mean exactly “can-be-static”, i.e., $-\vec{\partial} \hat{A}^0$ can survive for a static configuration, while the “dynamic” field $-\partial_t \vec{\hat{A}}$ cannot. This notation will prove quite illuminating for the non-Abelian field.] As promised, \vec{E}^{dy} and \vec{E}^{st} are separately gauge-invariant, since they are constructed with the gauge-invariant physical fields $\vec{\hat{A}}$ and \hat{A}^0 , respectively. Now the “static” and “dynamic” terms of the photon spin \vec{S}_γ can be defined as

$$\vec{S}_\gamma^{\text{st}} \equiv \int d^3x \vec{E}^{\text{st}} \times \vec{\hat{A}} = \int d^3x (-\vec{\partial} \hat{A}^0) \times \vec{\hat{A}} \quad (15a)$$

$$\vec{S}_\gamma^{\text{dy}} \equiv \int d^3x \vec{E}^{\text{dy}} \times \vec{\hat{A}} = \int d^3x (-\partial_t \vec{\hat{A}}) \times \vec{\hat{A}} \quad (15b)$$

Similarly, we can define “static” and “dynamic” terms of the photon orbital angular momentum:

$$\vec{L}_\gamma^{\text{st}} \equiv \int d^3x \vec{x} \times E_i^{\text{st}} \vec{\partial} \hat{A}_i = \int d^3x \vec{x} \times (-\partial_i \hat{A}^0) \vec{\partial} \hat{A}_i \quad (16a)$$

$$\vec{L}_\gamma^{\text{dy}} \equiv \int d^3x \vec{x} \times E_i^{\text{dy}} \vec{\partial} \hat{A}_i = \int d^3x \vec{x} \times (-\partial_t \hat{A}_i) \vec{\partial} \hat{A}_i \quad (16b)$$

The “static” terms $\vec{S}_\gamma^{\text{st}}$ and $\vec{L}_\gamma^{\text{st}}$ are not zero individually, but a little algebra shows that the sum $\vec{S}_\gamma^{\text{st}} + \vec{L}_\gamma^{\text{st}}$ always vanishes:

$$\vec{S}_\gamma^{\text{st}} = \int d^3x (-\vec{\partial} \hat{A}^0) \times \vec{\hat{A}} = \int d^3x \hat{A}^0 \vec{\partial} \times \vec{\hat{A}} \quad (17a)$$

$$\begin{aligned} \vec{L}_\gamma^{\text{st}} &= \int d^3x (-\partial_i \hat{A}^0) \vec{x} \times \vec{\partial} \hat{A}_i = \int d^3x \hat{A}^0 (\partial_i \vec{x}) \times \vec{\partial} \hat{A}_i + \int d^3x \vec{x} \times \vec{\partial} (\partial_i \hat{A}_i) \\ &= - \int d^3x \hat{A}^0 \vec{\partial} \times \vec{\hat{A}} = -\vec{S}_\gamma^{\text{st}} \end{aligned} \quad (17b)$$

Thus, it is not very meaningful to count $\vec{S}_\gamma^{\text{st}}$ as part of the photon spin and $\vec{L}_\gamma^{\text{st}} = -\vec{S}_\gamma^{\text{st}}$ as part of the photon orbital angular momentum, and we can wisely subtract these two canceling terms altogether from \vec{S}_γ and \vec{L}_γ . (This cancelation is also discussed by Wakamatsu [3].)

It is worthwhile to remark that the vanishing of “static” angular momentum is masked if one defines the total photon angular momentum as $\int d^3x \vec{x} \times (\vec{E} \times \vec{B})$. Similarly, by defining photon momentum as $\int d^3x \vec{E} \times \vec{B}$, one does not see the vanishing of a “static” term either, which however shows up clearly if one defines the photon momentum as $\vec{P}_\gamma \equiv \int d^3x E_i \vec{\partial} \hat{A}_i$:

$$\vec{P}_\gamma^{\text{st}} = \int d^3x E_i^{\text{st}} \vec{\partial} \hat{A}_i = \int d^3x (-\partial_i \hat{A}^0) \vec{\partial} \hat{A}_i = \int d^3x \hat{A}^0 \vec{\partial} (\partial_i \hat{A}_i) = 0. \quad (18)$$

Separation of the gluon spin \vec{S}_g , or essentially the non-Abelian color electric field \vec{E} , is much more involved due to non-linearity, but a satisfactory separation does exist:

$$\begin{aligned}\vec{E} &= -\partial_t \vec{A} - \vec{\nabla} A^0 + ig[\vec{A}, A^0] \\ &= -\vec{\mathcal{D}}_t \vec{A} - \vec{\mathcal{D}} \hat{A}^0 + ig[\vec{A}, \hat{A}^0] \\ &\equiv \vec{E}^{\text{dy}} + \vec{E}^{\text{st}} + \vec{E}^{\text{nl}}\end{aligned}\tag{19}$$

In arranging the first line into the second line, we have used the pure-gauge condition $\vec{E} \equiv -\partial_t \vec{A} - \vec{\nabla} A^0 + ig[\vec{A}, A^0] = 0$. In addition to the “dynamic” term $\vec{E}^{\text{dy}} \equiv -\vec{\mathcal{D}}_t \vec{A}$ and the “static” term $\vec{E}^{\text{st}} \equiv -\vec{\mathcal{D}} \hat{A}^0$, we find now a non-linear term $\vec{E}^{\text{nl}} \equiv ig[\vec{A}, \hat{A}^0]$. All these three terms are *individually* gauge-covariant. In consequence, we can separate the gluon spin \vec{S}_g into three gauge-invariant parts:

$$\vec{S}_g^{\text{st}} \equiv \int d^3x \vec{E}^{\text{st}} \times \vec{A} = \int d^3x (-\vec{\mathcal{D}} \hat{A}^0) \times \vec{A} \tag{20a}$$

$$\vec{S}_g^{\text{dy}} \equiv \int d^3x \vec{E}^{\text{dy}} \times \vec{A} = \int d^3x (-\vec{\mathcal{D}}_t \vec{A}) \times \vec{A} \tag{20b}$$

$$\vec{S}_g^{\text{nl}} \equiv \int d^3x \vec{E}^{\text{nl}} \times \vec{A} = \int d^3x ig[\vec{A}, \hat{A}^0] \times \vec{A} \tag{20c}$$

A similar gauge-invariant separation applies to the gluon orbital angular momentum:

$$\vec{L}_g^{\text{st}} \equiv \int d^3x \vec{x} \times E_i^{\text{st}} \vec{\mathcal{D}} \hat{A}_i = \int d^3x \vec{x} \times (-\vec{\mathcal{D}}_i \hat{A}^0) \vec{\mathcal{D}} \hat{A}_i \tag{21a}$$

$$\vec{L}_g^{\text{dy}} \equiv \int d^3x \vec{x} \times E_i^{\text{dy}} \vec{\mathcal{D}} \hat{A}_i = \int d^3x \vec{x} \times (-\vec{\mathcal{D}}_t \hat{A}_i) \vec{\mathcal{D}} \hat{A}_i \tag{21b}$$

$$\vec{L}_g^{\text{nl}} \equiv \int d^3x \vec{x} \times E_i^{\text{nl}} \vec{\mathcal{D}} \hat{A}_i = \int d^3x \vec{x} \times ig[\hat{A}_i, \hat{A}^0] \vec{\mathcal{D}} \hat{A}_i \tag{21c}$$

Although the expressions are much more complicated, we can still prove that the total “static” angular momentum $\vec{S}_g^{\text{st}} + \vec{L}_g^{\text{st}}$ vanishes identically. The easiest way to see this is to work in the Coulomb gauge, which gives $\vec{A}_\mu = 0$ and $\hat{A}_\mu = A_\mu$, thus \vec{S}_g^{st} and \vec{L}_g^{st} reduce to the same expressions as for the photon. Then, the same procedure as in Eqs. (17) shows $(\vec{S}_g^{\text{st}} + \vec{L}_g^{\text{st}})_{\vec{\partial} \cdot \vec{A}=0} = 0$. But since $(\vec{S}_g^{\text{st}} + \vec{L}_g^{\text{st}})$ is gauge invariant, it is identically zero in any gauge. The same reasoning shows that the “static” gluon momentum $\vec{P}_g^{\text{st}} \equiv \int d^3x E_i^{\text{st}} \vec{\mathcal{D}} \hat{A}_i \equiv 0$. But again, the vanishing of “static” gluon momentum and angular momentum are masked if they are defined through the Poynting vector $\vec{E} \times \vec{B}$.

Discarding the not-so-meaningful static term, we resume the 1-loop calculation and see how much “essential gluon spin” is left in a parent quark state. At 1-loop order the non-linear term \vec{S}_g^{nl} does not contribute. The remaining “dynamic gluon spin” \vec{S}_g^{dy} is found to

contribute

$$\begin{aligned}
S_g^{\text{dy}} &\equiv \langle p\sigma | \int d^3x (\vec{E}^{\text{dy}} \times \vec{\hat{A}})_3 | p\sigma \rangle \\
&= \langle p\sigma | \int d^3x (\vec{E}^{\text{dy}} \times \vec{\hat{A}})_3 | p\sigma \rangle_{\vec{\partial} \cdot \vec{\hat{A}}=0} \\
&= \frac{1}{5} S_g = \frac{1}{9} \Delta g = \sigma \cdot \frac{2}{9} \frac{\alpha_s}{\pi} \ln \frac{Q^2}{m^2}.
\end{aligned} \tag{22}$$

Quite remarkably, we see that S_g^{dy} is largely negligible compared to S_g or Δg , e.g., the specific renormalization found by choosing the ultraviolet cutoff Q^2 to be the same as the scale for $\alpha_s(Q^2)$ gives $S_g^{\text{dy}} \simeq 0.1\sigma$. To put this another way: the sizable “gluon spin”, S_g , is mostly a “static-field” contribution which is canceled exactly by the same “static-field” term in the “gluon orbital angular momentum”, L_g . *Significantly, this suggests that, defined properly, the gluon contribution to the nucleon spin may be drastically smaller than in the conventional wisdom.* In Ref.[2], a similar feature has been described for the gluon contribution to the nucleon momentum.

Although the Abelian and non-Abelian gauge fields share the same property that a “static” term can be shown to vanish for the total momentum and angular momentum, they do display a crucial difference as well: For the non-Abelian case, the non-linear field \vec{E}^{nl} can survive for a time-independent configuration, which therefore can possess non-trivial momentum and angular momentum. In this regard, it is useful to separate the non-Abelian \vec{E} into just two terms: $\vec{E} = \vec{E}^{\text{st}} + \vec{E}^{\text{nldy}}$, where $\vec{E}^{\text{nldy}} = \vec{E}^{\text{nl}} + \vec{E}^{\text{dy}}$ is the “non-linear-dynamic” field, and can be conveniently written as

$$\vec{E}^{\text{nldy}} = -\vec{\mathcal{D}}_t \vec{\hat{A}} + ig[\vec{\hat{A}}, \hat{A}^0] = -\mathcal{D}_t \vec{\hat{A}}. \tag{23}$$

Here $D_t = \partial_t + ig[A^0, \]$ is the complete (versus pure-gauge) covariant derivative. Accordingly, we can define the “non-linear-dynamic” gluon spin and orbital angular momentum as

$$\vec{S}_g^{\text{nldy}} \equiv \int d^3x \vec{E}^{\text{nldy}} \times \vec{\hat{A}} = \int d^3x (-\mathcal{D}_t \vec{\hat{A}}) \times \vec{\hat{A}} \tag{24a}$$

$$\vec{L}_g^{\text{nldy}} \equiv \int d^3x \vec{x} \times E_i^{\text{nldy}} \vec{\mathcal{D}} \hat{A}_i = \int d^3x \vec{x} \times (-\mathcal{D}_t \hat{A}_i) \vec{\mathcal{D}} \hat{A}_i \tag{24b}$$

These are the non-trivial quantities which do not cancel against each other. We can refine the decomposition in Eq. (9) to be

$$\vec{J}_{\text{QCD}} = \vec{S}_q + \vec{L}_q + \vec{S}_g^{\text{nldy}} + \vec{L}_g^{\text{nldy}}. \tag{25}$$

In closing, we comment on another possibly important application of separating \vec{E} into gauge-invariant/covariant pieces, regarding the construction of novel parton distribution functions. In Ref. [2], we have remarked how the separation $A_\mu \equiv \hat{A}_\mu + \bar{A}_\mu$ provides for a new definition of the polarized gluon parton distribution function (PDF) which corresponds to the gauge-invariant gluon spin, \vec{S}_g . Analogously, the further separation, $\vec{E} = \vec{E}^{\text{st}} + \vec{E}^{\text{nldy}}$, facilitates further the definition of a new PDF which corresponds to the “non-linear-dynamic gluon spin,” \vec{S}_g^{nldy} . One may perform the same study for the gluon momentum, but it is less significant because only the integrands are different in $\vec{P}_g \equiv \int d^3x E_i \vec{\mathcal{D}} \hat{A}_i = \int d^3x E_i^{\text{nldy}} \vec{\mathcal{D}} \hat{A}_i \equiv \vec{P}_g^{\text{nldy}}$.

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